Stable Populations

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A closed population subject to constant age-specific birth and death rates eventually becomes *stable,* with a constant growth rate and a constant age-distribution. We have already encountered the idea of a *stationary* population, where death rates are constant and there is a steady stream of births. In a stable population the birth stream growths exponentially over time. We review briefly some of the main ideas, focusing on the standard female dominant model. The textbook has an excellent discussion in Chapter 7.

The Renewal Equation

Let $B(t)$ denote the number of female births at time t, and let $N(a,t)$ be the number of women age a at time t. (These are both densities, so strictly speaking the number of births in a short interval of time is the product of $B(t)$ times the width of the interval.) Suppose that starting at time zero the age-specific birth and death rates become constant.

Let $m(a)$ denote the maternity function at any time after zero. If the reproductive span runs from ages α to β the number of births at time t is

$$B(t) = \int_{\alpha}^{\beta} N(a,t)m(a)da\,, \qquad t > 0$$

Between time zero and α the only women having children were already born at time zero, so they are products of whatever fertility and mortality regimes existed before. Between time α and β we have a mix of old timers and women who have been born in the new regime. But when we reach time β and beyond, all women in the reproductive ages have been born in the new regime; they are the product of the fertility and mortality schedules in the model.

Consider then $N(a,t)$, the number of women aged a at time t for $t > \beta$. These are the survivors of the cohort born at time $t - a$, which had initial size $B(t-a)$. Let $p(a)$ the probability of surviving to age a for someone born after time zero. We can then write

$$N(a,t) = B(t-a)p(a), \qquad t > \beta$$

and the stream of births becomes

$$B(t) = \int_{\alpha}^{\beta} B(t-a)p(a)m(a)da\,, \qquad t > \beta$$

This is an integral equation (an equation involving a function and its integral). We would like to solve it, by which we mean finding a function $B(t)$ that satisfies it. Lotka tried an exponential form, where

$$B(t) = Be^{rt}, \qquad t > 0$$

For $t > \beta$ we can also write $B(t-a) = Be^{r(t-a)} = Be^{rt}e^{-ra}$. If we substitute these results into the integral equation and cancel Be^{rt}, which appears on the left and right hand sides, we get after time β

$$1 = \int_{\alpha}^{\beta} e^{-ra} p(a) m(a) da$$

This is *Lotka's equation*. Note that instead of integrating from α to β we can integrate from 0 to t as the textbook does in equation 7.5. The integral from 0 to α is zero, and as long as $t > \beta$ the integral from β to t is also zero because $m(a)$ is zero outside the reproductive ages, so we only need to integrate over the reproductive ages. The textbook uses this fact in equation 7.10.

The next step is to see if this equation has a solution. Write ρ for r in the right-hand side and view that as a function of ρ. We'll assume that the survival and maternity schedules are well-behaved, so the integral is a continuous differentiable function of ρ. The function is always positive and declines monotonically as ρ increases, going from ∞ down to zero as ρ goes from $-\infty$ to $+\infty$. This means that there will be a value of ρ for which the function is one. This is Lotka's r, the *intrinsic growth rate.*

Estimating Lotka's r

Let us write the right hand side of Lotka's equation as a function of ρ

$$f(\rho) = \int_{\alpha}^{\beta} e^{-\rho a} p(a) m(a) da$$

We want a value of ρ such that $f(\rho) = 1$. The first derivative of this function w.r.t. ρ is

$$f'(\rho) = -\int_{\alpha}^{\beta} a\, e^{-\rho a} p(a) m(a) da$$

This derivative looks like a weighted mean age with weights $e^{-\rho a} p(a) m(a)$, except that we haven't divided by the sum of the weights, which is of course $f(\rho)$. Let us define the *mean age of childbearing*

$$A(\rho) = \frac{\int_{\alpha}^{\beta} a\, e^{-\rho a} p(a) m(a) da}{\int_{\alpha}^{\beta} e^{-\rho a} p(a) m(a) da}$$

We can then write the derivative as

$$f'(\rho) = -f(\rho) A(\rho)$$

This, by the way, shows that the function declines monotonically, as the derivative is always negative.

Coale proposed an iterative procedure for solving this equation. The method can be justified starting from a Taylor series expansion of $f(\rho)$ around the *solution*, where

$$f(\rho) \approx f(r) + (\rho - r) f'(r)$$

Solving for Lotka's *r* we obtain

$$r \approx \rho - \frac{f(\rho) - f(r)}{f'(r)}$$

Using the fact that $f'(r) = -A(r)f(r)$ and $f(r) = 1$ this equation simplifies to

$$r \approx \rho + \frac{f(\rho) - 1}{A}$$

where A is an approximation to the mean age of childbearing in the stable population, for example 27. This is the equation in Box 7.1 in the textbook.

An alternative direct application of Newton's method is to expand $f(r)$ around a trial value, so that

$$f(r) \approx f(\rho) + (r - \rho)f'(\rho)$$

Solving for r this equation and recalling that $f(r) = 1$ leads to

$$r \approx \rho + \frac{1 - f(\rho)}{f'(\rho)}$$

The two expansions are in fact equivalent but Newton uses the exact derivative at the trial value while Coale approximates it using an estimate of the derivative at the solution. Using the actual derivative often speeds convergence but, more importantly, yields the mean age of childbearing as a by-product.

Lotka himself used a quadratic approximation that requires no iteration but is less accurate than the iterative procedures.

In practice we need to use discrete data. With age groups of width n we will usually approximate the integral using midpoints and the usual survival and maternity functions, so

$$f(\rho) \approx \sum_{\alpha}^{\beta-n} e^{-\rho\left(x+\frac{n}{2}\right)} \frac{{}_nL_x}{l_0}\ {}_nF_x^F$$

and $f'(\rho) = -f(\rho)A(\rho)$, where the mean age of childbearing is estimated as

$$A(\rho) \approx \sum_{\alpha}^{\beta-n} \left(x + \frac{n}{2}\right) e^{-\rho\left(x+\frac{n}{2}\right)} \frac{{}_nL_x}{l_0}\ {}_nF_x^F / f(\rho)$$

(In both cases we are approximating the integral inside an age group $(x, x+n)$ by evaluating the integrand at the mid point $(x + n/2)$ and multiplying by the width of the interval n. The survival ratios are estimated as $p\left(x + \frac{n}{2}\right) = {}_nL_x/nl_0$. But the two n's cancel out, so I didn't show them.)

Box 7.1 in the textbook and the computing logs obtain an intrinsic r of 0.01424 for Egypt in 1977 after three iterations of Coale's method. The alternative procedure described here gives the same result and as a bonus gives the mean age of childbearing in the stable population, which is 29.47

The Stable Equivalent Age Distribution

Once we have an estimate of Lotka's r we can compute the stable age distribution. The population age a at time t for sufficiently large t (so that everyone has been born in the new regime) is

$$N(a,t) = B(t-a)p(a) = Be^{rt}e^{-ra}p(a)$$

The total population at time t can be obtained by integrating over all ages

$$N(t) = \int_0^\infty N(a,t)da = Be^{rt}\int e^{-ra}p(a)da$$

The proportion of the population age a at time t is then

$$c(a,t) = \frac{N(a,t)}{N(t)} = \frac{e^{-ra}p(a)}{\int e^{-rx}\,p(x)dx}$$

and *doesn't depend on* t, so we'll simply write $c(a)$.

We can simplify this a bit further if we think in terms of the instantaneous birth rate at time t, which is births divided by population:

$$b(t) = \frac{B(t)}{N(t)} = \frac{1}{\int e^{-ra}p(a)da}$$

where I have cancelled Be^{rt} in the numerator and denominator. The birth rate *doesn't depend on* t, so I will now write simply b. Moreover, the denominator of b is the same as the denominator of $c(a)$, so we can write

$$c(a) = b\,e^{-ra}p(a)$$

When working with discrete data we employ the usual mid-point approximations, so having obtained r we compute

$${}_nc_x = b\ e^{-r\left(x+\frac{n}{2}\right)}\,\frac{{}_nL_x}{l_0}$$

For the open-ended group one uses $x + e_x$ as the 'midpoint' and T_x instead of ${}_nL_x$ to approximate the integral. The birth rate is obtained as a normalizing constant such that the relative age distribution adds to one.

Box 7.2 in the textbook and the online computing logs calculate Lotka's r and the stable age distribution for U.S. females in 1991 using these procedures. Very similar results can be obtained from the first eigenvalue and eigenvector of the Leslie matrix. The online supplements also include a graph comparing the current and stable equivalent age distributions. The intrinsic growth rate is (slightly) negative, but the age structure is relatively young. As a result, we find that at 1991 rates the U.S. female population would have continued to growth for about 45 years before heading into extinction.

Why a Population Converges to Stability

My favorite proof of the basic theorem of stable population theory is due to Brian Arthur. His article is very clear and has the great merit of revealing the mechanism involved. While his proof is in discrete time, the gist of the argument can be conveyed equally well in continuous time.

Recall from page 1 that after time β the age distribution is given by

$$c(a,t) = \frac{B(t-a)p(a)}{\int_x B(t-x)p(x)dx}$$

Arthur notes that it is sufficient to show that the birth sequence eventually becomes exponential, say $B(t) \to Be^{rt}$, because if that is the case the age distribution becomes

$$c(a,t) = \frac{Be^{r(t-a)}p(a)}{\int_x Be^{r(t-x)}\,p(x)dx} = \frac{e^{-ra}p(a)}{\int_x e^{-rx}\,p(x)dx}$$

as Be^{rt} cancels out and we obtain an expression that doesn't depend on t!

The next insight comes from the observation that the birth sequence will become exponential if the ratio of $B(t)$ to Be^{rt} becomes a constant. Dividing the left and right-hand sides of the renewal equation by Be^{rt} we obtain

$$\frac{B(t)}{Be^{rt}} = \int_\alpha^\beta \frac{B(t-a)}{Be^{rt}} p(a)m(a)da$$

We can view the left-hand side as a growth-corrected birth sequence $\hat{B}(t) = B(t)/e^{rt}$. To obtain a similar expression on the right-hand side we multiply and divide by e^{-ra}, which leads us to

$$\hat{B}(t) = \int_\alpha^\beta \hat{B}(t-a)\, e^{-ra}p(a)m(a)da$$

So far the algebra holds for any value of r, but now we pick Lotka's r, which has the nice property that $\int e^{-ra}p(a)m(a)da = 1$. The reason why this is important is that we can now write

$$\hat{B}(t) = \int_\alpha^\beta \hat{B}(t-a)\, w(a)da$$

where $w(a)$ represents weights that integrate to one. In other words, the growth-corrected births at time t are a weighted average of the growth-corrected births in the past, where averaging is over a sliding window determined by the reproductive ages.

We can view this successive averaging as a form of smoothing because $\hat{B}(t)$, being a mean, is always *inside* the range of values between α and β years ago (unless they are all equal and the sequence has already converged). As time goes by and the window shifts we discard old values in favor of averages, until the range inevitable collapses and the sequence converges to a constant.