Population Momentum

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This unit focuses on population momentum, the notion that most of the world population would continue to grow even if fertility dropped suddenly to replacement level.

The Preston-Guillot Method

The textbook illustrates a method due to Preston and Guillot. The calculations are reproduced in the computing logs. We start from a maternity function ${}_nm_a$ and divide it by the NRR of 1.7028, assuming a proportionate decline to replacement level. We then compute the mean age of the net maternity schedule (new or old) which turns out to be 26.6. These quantities are computed as

$$NRR = {}_nL_x^F\, {}_nF_x^F \quad \text{and} \quad \bar{a} = \textstyle\sum_x (x + n/2)\, {}_nL_x^F\, {}_nF_x^F \;/NRR$$

The next step is to compute the weight function, representing the ratio of the expected number of births that would occur above (the mid-point of) each age to the mean age,

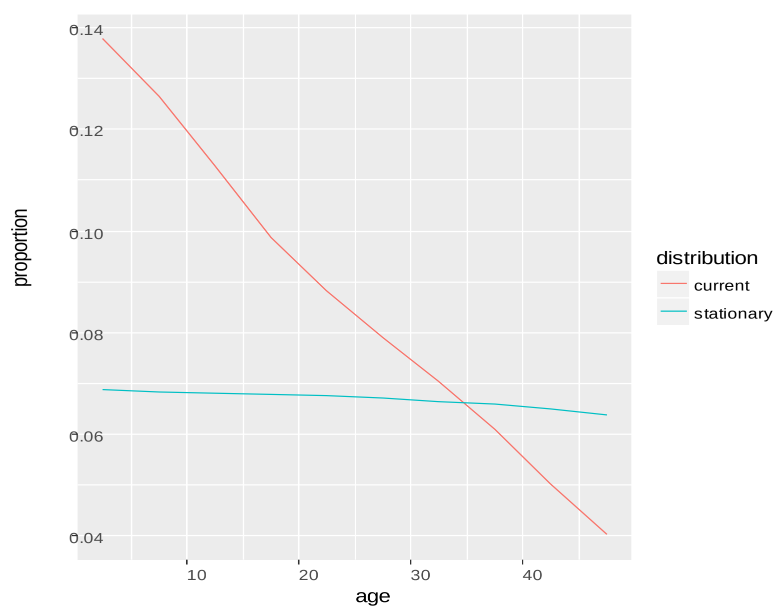
$${}_nw_x = (0.5\, {}_nL_x^F\, {}_nF_x^F \;+ \textstyle\sum_{x+5}\, {}_nL_a^F\, {}_nm_{ax})/\bar{a},$$

Finally we multiply by the ratio of the current and stationary equivalent age distributions and sum, so

$$M = \sum_a {}_nw_a\, {}_nc_a/\, {}_ns_a$$

where ${}_nc_a = {}_nN_a^F/N$ is the current age distribution and ${}_ns_a = {}_nL_a^F/e_0^F$ is the stationary equivalent.

We find that even if fertility dropped immediately to replacement level the female population would still growth 61%. The reason is apparent if we plot the current and stationary age distributions. The large proportions at young ages in the current age distribution compared to the stationary equivalent ${}_nL_a/e_0$ are the engine behind the momentum, particularly when weighted by the fact that most of their fertility is ahead of them.



Keyfitz's Approximation

The idea of population momentum originated with Keyfitz, and proved quite influential in policy circles. His formulation assumed that the population was stable at the outset and the reduction in rates was proportionally the same at all ages. The previous development doesn't require either assumption, and is therefore more general. Because Keyfitz's method is still popular, however, we apply it to Western Asia.

In the original 1971 paper momentum is given by

$$M = b\frac{e_0}{r\,m}\left(\frac{NRR-1}{NRR}\right)$$

where b is the birth rate, r is the rate of natural increase, m is the mean age of childbearing in the stationary population, and NRR is the net reproduction ratio, all before the fall to replacement level.

James Frauenthal pointed out that the formula was very nearly

$$M = b\frac{e_0}{\sqrt{NRR}}$$

and it is this simpler formula that is often used. (See Keyfitz and Caswell, pp. 197-198 for details.)

For our example the CBR is 0.029 and the formula gives $0.029 \times 79.3/\sqrt{1.7028} = 1.57$, which is pretty close to the more exact value of 1.61 computed above. Not bad for a simple calculation.

Reproductive Value

In class we provided some additional background on these results, starting from Fisher's notion of reproductive value. We discount future births at an annual rate r, so that the present value of a newborn 's future childbearing over the reproductive ages is

$$\int_\alpha^\beta e^{-ra}\,p(a)m(a)da$$

Setting the "interest rate" to Lotka's r gives a present value of 1. You may think of this as a newborn repaying her own life. The amount "owed" by a woman age x is known as her *reproductive* value and is

$$v(x) = \int_x^\beta e^{-r(a-x)}\,\frac{p(a)}{p(x)}m(a)da$$

Moving the terms on x out of the integral we can also write this as

$$v(a) = \frac{1}{e^{-rx}p(x)}\int_x^\beta e^{-ra}p(a)m(a)da$$

The Stable Equivalent Population

Each population has a *stable equivalent*, the population that would emerge if fertility and mortality stayed constant for a long time. That population has a growth rate r given by Lotka's r, intrinsic birth and death rates b and d, and a constant age structure $c(a)$, see equations 7.8, 7.9 and 7.10 in the textbook. What we don't know yet is its size. We will define the size as

$$Q = \lim_{t \to \infty} \frac{P_t}{e^{rt}}$$

One way to think about this construction is to project the population until it becomes stable and then reverse-project it at a constant rate r. The resulting population is stable and will eventually become indistinguishable from the target population, with the same size, age structure and vital rates.

Keyfitz shows that we can write the size of the stable equivalent population as

$$Q = \frac{\int_0^\beta n(x)v(x)dx}{b_r\, A_r}$$

where $n(x)$ is the female density at age x, $v(x)$ is Fisher's reproductive value, b_r is the birth rate and A_r is the mean age of childbearing in the stable population. This is an important formula because we can obtain the population momentum results from it, so let us write it in full glory as

$$Q = \int_0^\beta \frac{n(x)}{e^{-rx}p(x)} \int_x^\beta e^{-ra}p(a)m(a)da\, dx/(b_r A_r)$$

All we have done here is plug in Fisher's reproductive value.

Population Momentum

Suppose the maternity rates $m(a)$ were to change to replacement-level rates $m_0(a)$; for example we could set $m_0(a) = m(a)/\mathrm{NRR}$ changing all rates by the same proportion, but this restriction is not necessary. Suppose further that fertility and mortality then stay constant. Eventually the population will become stationary, and we can obtain its size from the general formula with $r = 0$

$$S = \int_0^\beta \frac{n(x)}{p(x)} \int_x^\beta p(a)m_0(a)da\, dx/(b_0 A_0)$$

where b_0 is the birth rate and A_0 the mean age of childbearing in the stationary population. Write $n(x) = Pc(x)$ where P and $c(x)$ are the current population size and age distribution, and note that $p(x) = c_0(x)/b_0$ where $c_0(x)$ is the stationary age distribution. Dividing both sides by P we then obtain momentum or S/P as

$$M = \int_0^\beta \frac{c(x)}{c_0(x)} \int_x^\beta p(a)m_0(a)da\, dx/A_0$$

This is the Preston-Guillot formula 7.21 in a slightly different guise, as taking A_0 inside the integral makes it their weight function $w(x)$. Note that they use a subscript s and a superscript * for the stationary population where I use a subscript 0. The main point here is that the mysterious weight function comes from Fisher's reproductive value.

Keyfitz Momentum

As noted earlier, Keyfitz's formula assumes that the population is already stable and fertility is reduced by the same proportion at all ages, so that $m_0(a) = m(a)/\text{NRR}$. We can obtain his result starting from the general formula. Write $n(x) = Pc_s(x)$ where $c_s(x)$ is the current stable age distribution and $p(x) = c_0(x)/b_0$ as we did before to obtain

$$M_k = \int_0^\beta \frac{c_s(x)}{c_0(x)} \int_x^\beta p(a)m(a)da\, dx/(A_0\text{NRR})$$

Recall that the stable age distribution is $c_s(x) = b_r e^{-rx}p(x)$ and the stationary is $c_0(x) = b_0 p(x)$ so the survival probabilities cancel. We can also write the stationary birth rate as $b_0 = 1/e_0$, so

$$M_k = \frac{b\, e_0}{A_0\text{NRR}} \int_0^\beta e^{-rx} \int_x^\beta p(a)m(a)da\, dx$$

Changing the order of integration and noting that $\int_0^a e^{-rx}dx = \frac{1}{r}(1 - e^{-ra})$ we obtain

$$M_k = \frac{b\, e_0}{rA_0\text{NRR}} \int_0^\beta (1 - e^{-ra})p(a)m(a)da$$

We then recognize the two terms in the integral as the NRR and Lotka's equation, so the result becomes

$$M_k = \frac{b\, e_0}{r\, A_0} \frac{\text{NRR} - 1}{\text{NRR}}$$

which is exactly Keyfitz's formula, writing A_0 instead of m for the stable mean age of childbearing.

Stable and Non-stable Momentum

Tom Espenshade and collaborators have a nice paper on population momentum in *Demography* in 2011 which I found very helpful. They call Keyfitz's formula the stable momentum, driven by differences between the stable and stationary age distributions, and Preston-Guillot's formula the total momentum, driven by differences between the current and stationary age distributions. They derive a formula for non-stable momentum, driven by differences between the current and stable age densities, and show that to a very close approximation total momentum is the product of stable and non-stable momentum.

In our example the Keyfitz formula worked well because most of the momentum in western Asia was stable. For the world as a whole, momentum around 2005 was 1.398, with 1.173 stable and 1.193 non-stable (for a product of 1.399), so the approximation would not be adequate. For the least developed countries, however, most of the momentum is stable (1.468 of 1.513).