Period Life Tables

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Rates to Probabilities

To convert a rate to a probability all we need is ${}_n a_x$, as we can then use

$$ {}_n q_x = \frac{n \quad {}_n m_x}{1 + (n - \quad {}_n a_x) \quad {}_n m_x} $$

This is the one and only formula you need to build a life table, everything else is pretty intuitive.

Coale and Demeny suggest values of ${}_n a_x$ for young ages, see Table 3.3 on page 48.

For the last open-ended age group we use

$$ {}_\infty a_x = \frac{1}{{}_\infty m_x} $$

or equivalently set ${}_\infty q_x = 1$.

Stationary Population

Interpreting a life table as a stationary population with l_0 births per year (see page 53):

- l_x is the number age x at last birthday
- ${}_n L_x$ is the number between ages x and $x + n$
- T_x is the population age x and above
- T_0 is the total population size
- ${}_n d_x$ is the number of deaths between ages x and $x + n$
- e_x is the mean age at death of people dying in a given year

What's the crude death rate? And the crude birth rate? The growth rate?

Mortality as a Continuous Process

- $l(x)$ is the number surviving to exact age x out of $l(0)$
- The death density $d(x)$ is

$$d(x) = \lim_{n\to 0} \frac{{}_nd_x}{n} = \lim_{n\to 0} \frac{l(x) - l(x+n)}{n} = -l'(x)$$

- The force of mortality $\mu(x)$ is

$$\mu(x) = \lim_{n\to 0} \quad {}_nm_x = \lim_{n\to 0} \frac{{}_nd_x}{{}_nL_x} = \lim_{n\to 0} \frac{{}_nd_x}{nl(x)} = \frac{d(x)}{l(x)} = -\frac{d}{dx}\log l(x)$$

- Integrating both sides and starting from $l(0)$ we get

$$l(x) = l(0)e^{-\int_0^x \mu(a)da}$$

The similarity to "the most important formula in demography" should not go unnoticed.

More on page 69

Probabilistic Interpretation

Let X denote a random variable representing age at death in a mortality regime given by $l(x)$

- Probability of surviving to age x is

$$\Pr(X > x) = \frac{l(x)}{l(0)}$$

- The density of age at death is

$$\lim_{dx\to 0} \frac{\Pr(X \in (x, x+dx))}{dx} = \frac{d(x)}{l(0)}$$

- The *conditional* density of age at death given survival to x is the force of mortality.

$$\lim_{dx\to 0} \frac{\Pr(X \in (x, x+dx) \mid X > x)}{dx} = \frac{d(x)}{l(x)} = \mu(x)$$

Rates to Probabilities Revisited

The probability of dying between ages x and $x+n$ conditional on having survived to x can be written as

$${}_nq_x = 1 - \frac{l(x+n)}{l(x)} = 1 - e^{-\int_x^{x+n} \mu(a)da}$$

If we assume that the force of mortality is *constant* between ages x and $x+n$ and estimate it using ${}_nm_x$ we get

$${}_nq_x = 1 - e^{-n\,{}_nm_x}$$

A simple formula for converting rates to probabilities. Note that ${}_na_x$ is not needed but is implicit. Solving for ${}_na_x$ in the usual formula for converting rates to probabilities we get

$$ {}_n a_x = n + \frac{1}{m} - \frac{n}{q} $$

and substituting the simple formula for ${}_n q_x$ we get the result on page 46.

Population and Life Table Rates

We now have some tools to have a closer look at these rates (see pages 61-62).

The life table mortality rates are

$$ {}_n m_x = \frac{\int_x^{x+n} l\,(a)\mu(a)da}{\int_x^{x+n} l\,(a)da} $$

a weighted average of the force of mortality in the age group with weights proportional to the stationary population.

The mortality rates observed in a population subject to the same force of mortality are

$$ {}_n M_x = \frac{\int_x^{x+n} N\,(a)\mu(a)da}{\int_x^{x+n} N\,(a)da} $$

where $N(a)$ is the population density at age a, so the rate is a weighted average of the force of mortality with weights proportional to the observed population.

There are only two cases in which equating these is right

1. When the population is stationary, so $n(a) \propto l(a)$, and
2. When the rates are constant within each age group.

This is one reason why I like the assumption of piecewise constant rates. In practice one hopes any differences within a single year of age will be too small to matter.